AN EXTREMAL PROBLEM IN BANACH SPACE WITH APPLICATIONS TO DISCRETE AND CONTINUOUS TIME OPTIMAL CONTROL

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1. INTRODUCTION

The theory of optimal control has received a new impetus through the papers of Gamkrelidze [1] and Neustadt [2]. It seems clear now that the optimal control problem should be studied as an extremal problem in a Banach space or a locally convex space. The motivation for this generality is derived from the study of optimal control problems with trajectory constraints. This author has arrived at the problem formulated in Section 3 through the study of nonlinear programming in general spaces [3]. The results obtained are similar to those of Neustadt, but the method of proof and the motivation appears to be different. It is hoped that this paper serves as a common framework for both optimal control and nonlinear programming problems.

[redacted]

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2. NOTATION, DEFINITIONS AND A PRELIMINARY RESULT

Throughout this paper, X and Y will denote arbitrary real Banach spaces. All undefined terms can be found in Dunford and Schwartz [4].

Def. 2.1. A function $f: X \to Y$ is differentiable (Fréchet-differentiable) at a point x if there is a continuous linear function, $f'(x)$, mapping X into Y such that

$$\lim_{\substack{\epsilon \to 0+ \\ w \to z}} \frac{f(x + \epsilon w) - f(x)}{\epsilon} = \langle f'(x), z \rangle \triangleq f'(x)(z)$$

$$\left(\lim_{h \to 0} \frac{f(x + h) - f(x) - \langle f'(x), h \rangle}{|h|} = 0 \right)$$

In addition to a linear approximation of a function at a point we shall need a 'linear' approximation of a set at a point.

Def. 2.2. Let A be an arbitrary subset of X and let $x \in A$. For each neighborhood N of x let $C(A \cap N, x)$ denote the smallest closed cone, with vertex 0, containing the set $A \cap N - x \triangleq \{z - x \mid z \in A \cap N\}$. Let $\mathcal{N}$ be the neighborhood system at x. Then the set

$$LC(A, x) \triangleq \cap \{C(A \cap N, x) \mid N \in \mathcal{N}\}$$

is called the local cone of A at x.

Def. 2.3a. Let A be an arbitrary subset of X and $x \in A$. The set

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$$LP(A, x) \triangleq \left\{ x^* \in X^{*\,[1]} \,\middle|\, \left\langle x^*, z \right\rangle \le 0 \text{ for all } z \in LC(A, x) \right\}.$$

in X^* is called the local polar of A at x.

Def. 2.3b. If K is a cone then $P(K) \triangleq LP(K, 0)$.

Remark 2.1a. The local cone is a nonempty (it always contain 0) closed cone and the local polar is a nonempty closed convex cone. b. A useful alternative characterization of the local cone is given by the next fact.

Fact 2.1. The following statements are equivalent. a. $z \in LC(A, x)$. b. There exist sequences $\{x_n\} \subseteq A$, $\{\lambda_n\}$, $\lambda_n > 0$ such that, $x_n \to x$ and $\lambda_n(x_n - x) \to z$. c. There exist sequences $\{z_n\} \subseteq X$, $\{\epsilon_n\}$, $\epsilon_n > 0$, such that $\epsilon_n \to 0$, $z_n \to z$ and $(x + \epsilon_n z_n) \in A$.

Proof. Trivially b. and c. are equivalent. The equivalence of a. and b. follows directly from Def. 2.2 using a standard Cantor diagonal argument. Q.E.D.

The justification of the two linear approximations is provided by the following elementary but extremely useful result.

Theorem 2.1. Let f be a real-valued function of x and A an arbitrary subset of X. Let x in A be a solution (2.1)

(2.1) $\quad \mathrm{Max}\{f(x) \mid x \in A\}$

[1] X^* denotes the space of all real-valued, continuous linear functions on X.

Then, if f is differentiable (see Def. 2.1) at x we must have

(2.2) $\qquad f'(x) \in LP(A, x)$

Proof. Let $z \in LC(A, x)$. We have to show that $\langle f'(x), z \rangle \leq 0$. By Fact 2.1c there are sequences $z_n \to z$, $\epsilon_n \to 0+$ such that $x_n = (x + \epsilon_n z_n) \in A$. Since x solves (2.1), $f(x_n) - f(x) \leq 0$. Hence,

$$\frac{f(x + \epsilon_n z_n) - f(x)}{\epsilon_n} \leq 0$$

Taking the limit as $n \to \infty$, we get (2.2) from Def. 2.1.

Q.E.D.

Remark 2.2. The definitions of derivative, local cone and local polar make sense for arbitrary linear topological spaces. Fact 2.1 is valid if we replace 'sequence' by 'generalized sequence' or 'net'. Theorem 2.1 still remains true.

3. STATEMENT OF THE MAIN THEOREM AND SOME COMMENTS

Theorem 3.1. Let X and Y be real Banach spaces. Let f be a real-valued differentiable function of x, and g, a continuously Fréchet-differentiable function from X to Y. Let A be a subset of X and suppose that $\underline{x}$ solves (3.1)

(3.1) $\qquad \mathrm{Max}\{f(x) \mid g(x) = 0, \ x \in A\}$

Let $G \equiv g'(\underline{x})$ be the derivative of g at $\underline{x}$. Let K_1 be any closed convex cone contained in $LC(A, \underline{x})$. Then if G and K_1 satisfy assumptions A1 and A2 there exists a number $\mu \geq 0$ and a y^* in Y^* not both zero such that

$$(3.2) \qquad \langle \mu f'(\underline{x}), \delta x \rangle + \langle y^*, G(\delta x) \rangle \leq 0 \quad \text{for all } \delta x \text{ in } K_1 .$$

A1. Suppose $G(K_1) = Y$ and let $z \in K_1$, $z \neq 0$. Then we shall assume that there is a closed convex cone K, depending on z and contained in K, which satisfies the following conditions: 1. $G(K) = Y$. 2. There exists a closed linear subspace Z of X containing K such that K has a nonempty interior K_0 relative to Z and $z \in K_0$. 3. Finally if $z(\epsilon)$ for $\epsilon > 0$ is an arc in K_0 such that $z(\epsilon) \to 0$ and z is differentiable from the right at $\epsilon = 0$ with $z'(0) = z$, then there is a sequence $\epsilon_n \to 0$ such that $(\underline{x} + z(\epsilon_n))$ is in A for each n.

A2. 1. If $\overline{G(K_1)} = Y$, then we assume that $G(K_1) = Y$. 2. Let $N = \{x \mid G(x) = 0\}$. We will assume that $LP(N) + LP(K_1)$ is closed.

Comments. The assumptions A2 are of a technical nature and in most problems they are satisfied. In most applications of discrete and continuous optimal control the range space Y is finite-dimensional. In this case, it can be easily shown that these assumptions are automatically satisfied.

The assumptions A1 are far more serious, and can be considered as compatibility requirements at the optimal point, between the function

g, the set A and their 'linear' approximations G and K_1. As is shown in section 5 these requirements are satisfied by most optimal control problems. See also [1, 2, 3].

The requirement of the strong differentiability of g can be replaced by the weaker notion of differentiability if Y is finite-dimensional. The only place the stronger notion is employed is in Lemma 2 of the Appendix. It is probable, although the author is unable to prove it, that this result is valid with only the weaker notion of differentiability.

4. PROOF OF THE MAIN THEOREM

The proof is divided into two parts; the first case takes care of the degeneracies which may arise, the second case is the important one.

Case 1. Let $Q \triangleq \overline{G(K_1)}$. Suppose $Q \neq Y$. Then Q is a proper closed convex cone in Y so that there is a y^* in Y^*, $y^* \neq 0$ such that

$$\langle y^*, \delta y \rangle \leq 0 \quad \text{for all } \delta y \text{ in } Q$$

$$\therefore \langle y^*, G(\delta x) \rangle \leq 0 \quad \text{for all } \delta x \text{ in } K_1.$$

Hence Equation (3.2) is satisfied with $\mu = 0$ and $y^* \neq 0$.

Case 2. Suppose $Q \triangleq \overline{G(K_1)} = Y$. Then by assumption A2, (4.1) $$G(K_1) = Y$$

Let $A_g \triangleq \{x \mid g(x) = 0\}$ and let $N \triangleq \{x \mid G(x) = 0\}$. We will now prove the important fact that

(4.2) $$LC(A_g \cap A, \underline{x}) \supseteq K_1 \cap N$$

Let $z \in K_1 \cap N$ and suppose $z \neq 0$. By assumption A1, there exists a closed convex cone $K \subseteq K_1$ which satisfies the following conditions: 1. $G(K) = Y$. 2. There is a closed linear subspace Z of X containing K such that K has a nonempty interior K_0 relative to Z and z is in K_0. By the corollary to Lemma 2 of the Appendix there exists an arc $z(\epsilon)$, $\epsilon > 0$, contained in K_0 such that it is differentiable from the right at $\epsilon = 0$ and such that

$$\lim_{\epsilon \to 0} z(\epsilon) = 0, \quad z'(0) = z$$

and

(4.3) $$g(\underline{x} + z(\epsilon)) = 0 \text{ for each } \epsilon$$

But then by assumption A1, there is a sequence $\epsilon_n \to 0$ such that, $(\underline{x} + z(\epsilon_n))$ is in A for each n. Because of (4.3) we see that

(4.4) $$(\underline{x} + z(\epsilon_n)) \in \{A_g \cap A\} \quad \text{for each } n.$$

But by Fact 2.1c Equation (4.3) implies that

$$z \in LC(A_g \cap A, \underline{x})$$

which proves the assertion (4.2). Directly from the definition of the local polar (4.2) implies that,

$$\text{(4.5)} \qquad LP(A_g \cap A, \underline{x}) \subseteq P(K_1 \cap N).$$

Since $\underline{x}$ is a solution of the problem (3.1), Theorem 2.1 says that

$$\text{(4.6)} \qquad f'(\underline{x}) \in LP(A_g \cap A, \underline{x}) \subseteq P(K_1 \cap N).$$

It is straightforward to show [3, p. 12], using the strong separation theorem [4, p. 417] that

$$\text{(4.7)} \qquad P(K_1 \cap N) = \overline{P(K_1) + P(N)}$$

By assumption A2 $P(K_1) + P(N)$ is closed, so that (4.6) and (4.7) give,

$$\text{(4.8)} \qquad f'(\underline{x}) \in P(K_1) + P(N)$$

By ([1], p. 487), using Equation (4.1) and Def. 2.3b we obtain,

$$\text{(4.9)} \qquad P(N) = \{Y^* \cdot G\} = \{y^* \cdot G \mid y^* \in Y^*\}$$

where $y^* \cdot G$ is the element in X^* given by $\langle y^* \cdot Gx \rangle \equiv \langle y^*, Gx \rangle$. From (4.8) and (4.9) we see that there is a y^* in Y^* such that

$$(f'(x) + y^* \cdot G) \in P(k_1)$$

Hence (3.2) is again satisfied with $\mu = 1$, and the proof is completed.

Q.E.D.

5. APPLICATION OF THEOREM 4.1

A. Discrete Optimal Control

Consider a difference equation,

$$x(k+1) = x(k) + f(x(k), u(k)) \quad k = 0, 1, \ldots$$

where $x \in X$ is the state vector, $u \in U$ is the control vector and $f : X \times U \to X$ is a continuously Fréchet-differentiable function. X and U are arbitrary B-spaces. Let n be a fixed integer representing the duration of the process. Let A_0 and A_n be subsets of X representing the initial and target set respectively. Let $\Omega \subseteq U$ be the set of available controls. The gain function g is a real-valued differentiable function on $X^n \times U^{n-1}$. We are required to

$$\text{(5.1)} \qquad \operatorname{Max}\{g(x(0), \ldots, x(N); u(0), \ldots, u(n-1)\}$$

subject to

$$\text{(5.2)} \qquad h(x(k+1), x(k), u(k)) = x(k+1) - x(k) - f(x(k), u(k)) = 0$$

$$\text{for } 0 \leq k \leq n-1$$

and

$$\text{(5.3)} \qquad x(0) \in A_0, \; x(n) \in A_n, \; u(k) \in \Omega \quad \text{for } 0 \leq k \leq n-1$$

Let $\{\underline{u}(0), \ldots, \underline{u}(n-1)\}$ be the optimal control and $\{\underline{x}(0), \ldots, \underline{x}(n)\}$ be the optimal trajectory. Let K_0, K_n be closed convex cases contained in $LC(A_0, \underline{x}(0))$ and $LC(A_n, \underline{x}(n))$ respectively. Let Q_i be a closed convex cone contained in $LC(\Omega, \underline{u}(i))$ for $0 \leq i \leq n-1$. Now we form the function,

$$\Phi(\mu; x(0), \ldots, x(n); u(0), \ldots, u(n-1); \psi(1), \ldots, \psi(n))$$
$$= \mu g_n(x(0), \ldots, x(n); u(0), \ldots, u(n-1) + \sum_{k=0}^{n-1}$$
$$\left\langle \psi(k+1), h(x(k+1), x(k), u(k)) \right\rangle$$

where the $\psi(k)$ belong to X^*.

Suppose the cones defined above, the function h and the constraint sets satisfy the assumptions of Theorem 4.1. Then there exists $\mu = \underline{\mu} \geq 0$, $\psi(k) = \underline{\psi}(k)$ not all zero such that

$$\left\langle \frac{\partial \Phi}{\partial x(0)}, \delta x \right\rangle \leq 0 \quad \text{for } \delta x \in K_0.$$

$$\frac{\partial \Phi}{\partial x(k)} = 0 \quad \text{for } 0 < k \leq n-1.$$

$$\left\langle \frac{\partial \Phi}{\partial x(n)}, \delta x \right\rangle \leq 0 \quad \text{for } \delta x \in K_n.$$

$$\left\langle \frac{\partial \Phi}{\partial u(k)}, \delta x \right\rangle \leq 0 \quad \text{for } \delta u \in \Omega_k, \; 0 \leq k \leq n-1.$$

where the derivatives are evaluated at $\mu = \underline{\mu}$, $x(k) = \underline{x}(k)$, $u(k) = \underline{u}(k)$ and $\psi(k) = \underline{\psi}(k)$. Now if we expand the above equations we obtain the usual necessary conditions for discrete optimal control.

Remarks. 1. The conditions given in [5] are a special case of the above equations. 2. The fact that we allow our state variables to be infinite-dimensional will also enable us to consider discrete stochastic optimal control problems. See [3] for an elementary example.

B. Continuous-Time Optimal Control

Let $\mathscr{F}$ be the linear space whose elements $f(x, t)$ are n-dimensional real vector-valued functions for x in R^n and t in a fixed finite closed interval $I = [t_0, t_1]$. The functions f satisfy certain smoothness conditions in x and some integrability conditions in t. Let F be a quasi-convex subset of $\mathscr{F}$. For the precise conditions and definition the reader is referred to Gamkrelidze [1] and Neustadt [2]. The relevance of the various assumptions made in the sequel to optimal control problems is also discussed in these references.

Now for any f in F, let $x(t)$, t in I be any absolutely continuous solution of the differential equation

$$\dot{x}(t) = f(x(t), t), \quad t \text{ in } I \tag{5.4}$$

We shall regard such a function x as an element of the Banach space X of all continuous functions from the compact interval I into R^n. We also define A to be the set consisting of those elements x in X which are solutions of (5.4) for some f in F. Now let h be a real-valued differentiable function of x in X and let $g: X \rightarrow R^m$ be continuouly differentiable function. We wish to solve the following problem:

(5.5) $$\operatorname{Max}\{h(x) \mid g(x) = 0,\ x \in A\}$$

Let $\underline{x}$ be a solution of (5.5) and suppose that

(5.6) $$\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t),\ t),\quad t \text{ in } I$$

for some $\underline{f}$ in F. Let $[F]$ denote the convex hull of F, and consider the linear variational equation of (5.6),

(5.7) $$\delta\dot{x}(t) = \frac{\partial f}{\partial x}(\underline{x}(t),\ t)\ \delta x(t) + \Delta f(\underline{x}(t),\ t)$$

for t in I. Here Δf is any arbitrary element of the set $\{[F] - \underline{f}\}$ and $\delta x(t_0) = \xi$ is any arbitrary n-vector. Let $\varphi(t)$ be a non-singular matrix solution of the homogeneous matrix differential equation

$$\dot{\varphi}(t) = \frac{\partial f}{\partial x}(\underline{x}(t),\ t)\ \varphi(t)$$

with $\varphi(t_0) = 1$, the identity matrix. Then the solution of (5.7) is

(5.8) $$\delta x(t) = \varphi(t)\left\{\xi + \int_{t_0}^{t} \varphi^{-1}(t)\ \Delta f(\underline{x}(t),\ t)\ dt\right\}$$

Let $K \subseteq X$ be the collection of all δx which satisfy (5.8) for some ξ in R^n and some function Δf in $\{[F] - \underline{f}\}$. Clearly K is convex and let K_1 be the closed convex cone generated by K. Using the definition of quasi-convexity and the (generalized) Gronwall's lemma [6] it is easy to show that $K \subseteq LC(A,\ \underline{x})$. We therefore have

<u>Lemma 5.1.</u> F is quasi-convex $\Rightarrow K_1 \subseteq LC(A, \underline{x})$.

In order to apply Theorem 3.1 we have to verify that assumptions A1 and A2 are satisfied. First of all since the range of g is finite-dimensional, A2 is automatically satisfied. Let G be the derivative of g at the optimal point $\underline{x}$, and suppose that $G(K_1) = R^m$. Let $z \in K_1$, $z \neq 0$ and $G(z) = 0$. Let Σ be a simplex in R^m, generated by the points, $y_0, \ldots, y_m$ containing 0 in its interior. Let $k_0, \ldots, k_m$ be in K_1 such that $G(k_i) = y_i$ for $0 \leq i \leq m$. Let K be the polyhedral cone generated by $k_0, \ldots, k_m$. Using the definition of quasi-convexity and the (generalized) Gronwall's lemma it can be shown that K satisfies assumption A1. Then by Theorem 3.1 there exists numbers $\mu \geq 0$, $\lambda_1, \ldots, \lambda_m$ not all zero such that,

$$(5.9) \qquad \mu \left\langle f'(\underline{x}), \delta x \right\rangle + \left\langle \lambda, G(\delta x) \right\rangle \leq 0 \text{ for all } \delta x \text{ in } K_1.$$

where $\lambda = (\lambda_1, \ldots, \lambda_m)$. Following Neustadt [2] we can obtain the maximum principle, from Equation (5.9).

<u>Remarks.</u> Theorem 3.1 deals with a problem which may have infinitely many constraints(since X and Y are arbitrary Banach spaces). However in the above application we have only considered finitely many constraints since $Y = R^m$. It appears to the author that the notion of quasi-convexity is too weak in that, generally, assumption A1 will not be satisfied for any arbitrary Banach space Y. If F is convex instead of quasi-convex, these conditions usually hold.

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APPENDIX: PROOF OF LEMMA 2

We shall prove two results which are of independent interest and which are also required to complete the proof of Theorem 3.1.

Lemma 1. Let X and Y be real B-spaces and let G be a continuous linear mapping from X into Y. Let K be a closed convex cone in X such that $G(K) = Y$.

For each $\rho > 0$, let $K_\rho \triangleq \{\delta x \mid |\delta x| < \rho, \ \delta x \in K\}$. Then there is a number $m > 0$, independent of ρ, such that

$$G(K_\rho) \supseteq S_{m\rho}$$

where $S_{m\rho}$ is the closed sphere in Y of center 0 and radius $m\rho$.

Proof. This result is a generalization of the Interior Mapping Principle. Although the proof is long, it is a straightforward modification of that given by Dunford and Schwartz*. Hence the proof is omitted.

Q.E.D.

Lemma 2. Let K be a closed convex cone in X, and g, a continuously Fréchet-differentiable function from X to Y such that $g(0) = 0$. Let $G \equiv g'(0)$ and suppose that there is a number $m > 0$ such that for $\rho > 0$, $G(K_\rho) \supseteq S_{m\rho}$. Let $z \in K$, $|z| = 1$, and $G(z) = 0$. Then there exists a number $\epsilon_0 > 0$, and a function $o(\epsilon)$ such that for all $0 < \epsilon < \epsilon_0$, the set $g(\epsilon z + K_{o(\epsilon)})$ is a neighborhood of 0 in Y.

* Dunford and Schwartz, Linear Operators Part I, pp. 55-56.

Proof. Let $v: X \to Y$ be the function defined by $v(x) = g(x) - G(x)$. Then,

$$|v(\epsilon z + x_1) - v(\epsilon z + x_2)|$$

$$= |g(\epsilon z + x_1) - g(\epsilon z + x_2) - G(x_1 - x_2)|$$

$$= |\langle g'(\epsilon z + x_1), x_1 - x_2 \rangle + o_1(|x_1 - x_2|) - G(x_1 - x_2)|$$

Therefore,

$$\frac{|v(\epsilon z + x_1) - v(\epsilon z + x_2)|}{|x_1 - x_2|} \leq \|g'(\epsilon z + x_1) - G\| + \frac{o_1(|x_1 - x_2|)}{|x_1 - x_2|}$$

Also,

$$|v(\epsilon z + x_1)| = |g(\epsilon z + x) - G(\epsilon z + x)| = o_2(|\epsilon z + x|)$$

Pick a number $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$,

$$\frac{|v(\epsilon z + x_1) - v(\epsilon \dot{z} + x_2)|}{|x_1 - x_2|} < \frac{m}{4} \quad \text{for } |x_i| < \epsilon \quad i = 1, 2.$$

and

$$o_2(|\epsilon z + x|) \triangleq o(\epsilon) < \frac{m}{4} \quad \text{for } |x| < \epsilon.$$

Fix $0 < \epsilon < \epsilon_0$ and let $y \in Y$ with $|y| < o(\epsilon)$.

Let $x_0 \in K$ such that $G(x_0) = y$ and $|x_0| < \frac{1}{m} |y| < \frac{1}{m} o(\epsilon)$.

Let $x_1 \in K$ such that $G(x_1 - x_0) = -v(\epsilon z + x_0)$ and $|x_1 - x_0| < \frac{1}{m} |v(\epsilon z + x_0| < \frac{1}{m} o(\epsilon)$.

For $n \geq 1$, let $x_{n+1} \in D$ with $G(x_{n+1} - x_n) = -v(\epsilon z + x_n) + v(\epsilon z + x_{n-1})$ and $|x_{n+1} - x_n| < \frac{1}{m} o(\epsilon)$.

We first show that for $n \geq 0$, $|x_n| < \epsilon$ so that the above inequalities are valid. Firstly,

$$|x_0| < \frac{1}{m} o(\epsilon) < \frac{1}{4}\epsilon \quad \text{and} \quad |x_1 - x_0| < \frac{1}{m} o(\epsilon) < \frac{1}{4}\epsilon$$

$$\therefore |x_1| \leq |x_0| + |x_1 - x_0| < \frac{1}{2}\epsilon .$$

By induction on n,

$$|x_{n+1} - x_n| < \left(\frac{o(\epsilon)}{m}\right)^n |x_1 - x_0| < \left(\frac{1}{4}\right)^n |x_1 - x_0|$$

$$\therefore |x_{n+\rho} - x_n| < \left(\frac{1}{4}\epsilon\right)^\rho \frac{1}{1 - \frac{1}{4}\epsilon} |x_1 - x_0| < \left(\frac{1}{4}\epsilon\right)^\rho \frac{2}{m} o(\epsilon).$$

In particular, $|x_{n+1} - x_1| < \frac{\epsilon}{2}$ so that $|x_{n+1}| < \frac{4}{m} o(\epsilon) < \epsilon$. Also x_n converges. Let $\lim x_n = x$. Then $|x| < \frac{4}{m} o(\epsilon)$ and $x \in K$. Now,

$$G(x_0) = y$$

$$G(x_1) - G(x_0) = -v(\epsilon z + x_0)$$

$$G(x_2) - G(x_1) = -v(\epsilon z + x_1) + v(\epsilon z + x_0)$$

$$\vdots$$

$$G(x_{n+1}) - G(x_n) = -v(\epsilon z + x_n) + v(\epsilon z + x_{n-1}).$$

Adding both sides we get,

$$G(x_{n+1}) = y - v(\epsilon z + x_n) \quad \text{for } n \geq 0$$

$$y = G(x_{n+1}) - v(\epsilon z + x_n)$$

Also $|y - g(\epsilon z + x_n)| = |y - G(x_n) - v(\epsilon z + x_n)|$

$$= |y - G(x_n) + G(x_{n+1}) - G(x_{n+1}) - v(\epsilon z + x_n)|$$

$$= |G(x_{n+1} - x_n)| \leq \|G\| \; |x_{n+1} - x_n| \to 0 \text{ as } n \to \infty.$$

But $x_n \to x$ so that $g(\epsilon z + x) = y$. Also $x \in D$ and $|x| < \frac{4}{m} o(\epsilon)$.
Therefore,

$$g(\epsilon z + K_{o(\epsilon)}) \supseteq S_{(m/4)o(\epsilon)} \qquad \text{Q.E.D.}$$

Corollary. Let $g: X \to Y$ be a continuously Fréchet-differentiable function with $g(\underline{x}) = 0$. Let $G \triangleq g'(\underline{x})$. Let K be a closed convex cone in X with $G(K) = Y$, and let Z be a closed linear subspace of X such that K has nonempty interior K_0 relative to Z. Let $z \in K_0$ with $G(z) = 0$. Then there exists an arc $z(\epsilon)$, $\epsilon > 0$ in K_0 such that

1) $z(\epsilon) \to 0$ as $\epsilon \to 0$
2) $z(\epsilon)$ is differentiable from the right at $\epsilon = 0$ with $z'(0) = z$ and
3) $g(\underline{x} + z(\epsilon)) = 0$ for all ϵ.

Proof. By Lemmas 1 and 2 there exists a function $o(\epsilon)$ such that $g(\underline{x} + \epsilon z + K_{o(\epsilon)})$ is a neighborhood of 0 in Y.

Then for $\epsilon > 0$ there exists a vector $x(\epsilon)$ in K with $|x(\epsilon)| < o(\epsilon)$ such that $g(\underline{x} + \epsilon z + x(\epsilon)) = 0$. Define $z(\epsilon) = \epsilon z + x(\epsilon)$. The rest follows.

Q.E.D.

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